$$Tr(F_B) = Tr(F_{B'}) = \dots = \sum_{\forall i} \lambda_i$$

$$|F_B| = |F_{B'}| = \dots = \prod_{\forall i} \lambda_i$$

Trace and determinant of a matrix of an Endomorphism remain ctt. no matter the base.

e.g.

$$f(\bar{x}) = (x^1 + x^2, x^1 + x^2, x^3)$$

$$B = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$$

$$F_B = \begin{array}{c} \\ \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \begin{array}{ccc} f(\bar{e}_1) & f(\bar{e}_2) & f(\bar{e}_3) \end{array} \\ \text{in } B \end{array}$$

$$Tr(F_B) = 1 + 1 + 1 = 3$$

$$|F_B| = 0$$

$$\left.\begin{array}{l} \bar{u}_1 = (1, -1, 0)_B \in S(0) \\ \bar{u}_2 = (0, 0, 1)_B \in S(1) \\ \bar{u}_3 = (1, 1, 0)_B \in S(2) \end{array}\right\} B' = [\bar{u}_1, \bar{u}_2, \bar{u}_3]$$ Base of Eigenvectors

$$F_{B'} = \begin{array}{c} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \\ \begin{array}{ccc} f(\bar{u}_1) & f(\bar{u}_2) & f(\bar{u}_3) \end{array} \\ \text{in } B' \end{array}$$

$$Tr(F_{B'}) = 0 + 1 + 2 = 3$$

$$|F_{B'}| = 0$$

Eigenvalues 2

A certain endomorphism f in $\mathbb{R}^3$ has two different eigenvalues with a different sign each.
The determinant for the matrix of f is 1 while the trace of the matrix is -1.
We also know that:
· ker(f+1)=L{(1,1,0),(0,1,1)}
· (1,1,1) is an eigenvector
Determine:
1. F_B ?
2. F_B^{102}?

$$\boxed{S(\lambda)=Ker(f-\lambda i)}$$

$$\text{Eigenvalues}\begin{cases}\lambda_1 \longrightarrow MO(\lambda_1)=2\\ \lambda_2 \longrightarrow MO(\lambda_2)=1\end{cases}$$

$$sg(\lambda_1)\neq sg(\lambda_2)$$

$$\left.\begin{array}{l}|F_B|=1=\lambda_1\lambda_1\lambda_2\\ Tr(F_B)=-1=\lambda_1+\lambda_1+\lambda_2\end{array}\right\}\begin{array}{l}\lambda_1<0\\ \lambda_2>0\end{array}$$

$$Ker(f+1)=\mathcal{L}\{\overbrace{(1,1,0)}^{\bar{u}_1},\overbrace{(0,1,1)}^{\bar{u}_2}\}=S(-1)$$

$$\text{If } (1,1,1) \text{ is an eigenvector AND } \begin{vmatrix}1&1&0\\0&1&1\\1&1&1\end{vmatrix}\neq 0 \longrightarrow S(1)=\mathcal{L}\{\overbrace{(1,1,1)}^{\bar{u}_3}\}$$

$$B'=\{\underbrace{\bar{u}_1,\bar{u}_2}_{\in S(-1)},\underbrace{\bar{u}_3}_{\in S(1)}\} \text{ Base of Eigenvectors}$$

$$F_{B'}=\begin{array}{c}\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix}\\ f(\bar{u}_1)\ f(\bar{u}_2)\ f(\bar{u}_3)\end{array}$$

One way of getting F_B:

$$F_{B'} = \begin{array}{c} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ f(\bar{u}_1)\, f(\bar{u}_2)\, f(\bar{u}_3) \end{array} \qquad B' \begin{cases} \bar{u}_1 = (1,1,0)_B \\ \bar{u}_2 = (0,1,1)_B \\ \bar{u}_3 = (1,1,1)_B \end{cases}$$

$$C = \begin{array}{c} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \\ \bar{u}_1 \ \bar{u}_2 \ \bar{u}_3 \\ \text{in } B \end{array}$$

$$\boxed{F_{B'} = C^{-1} F_B C} \Rightarrow F_B = C F_{B'} C^{-1}$$

if $\bar{u}_1 \in S(-1)$ $\quad f(\bar{u}_1) = -\bar{u}_1 = (-1,0,0)_{B'}$

if $\bar{u}_2 \in S(-1)$ $\quad f(\bar{u}_2) = -\bar{u}_2 = (0,-1,0)_{B'}$

if $\bar{u}_3 \in S(1)$ $\quad f(\bar{u}_3) = \bar{u}_3 = (0,0,1)_{B'}$

$$C^{-1} = \frac{Adj(C)^t}{|C|}$$

$$\begin{cases} \bar{u}_1 = \bar{e}_1 + \bar{e}_2 & (1) \\ \bar{u}_2 = \bar{e}_2 + \bar{e}_3 & (2) \\ \bar{u}_3 = \bar{e}_1 + \bar{e}_2 + \bar{e}_3 & (3) \end{cases}$$

$\bar{u}_3 - \bar{u}_1 = \bar{e}_3 \quad (3)-(1)$

$\bar{u}_3 - \bar{u}_2 = \bar{e}_1 \quad (3)-(2) = (4)$

$\bar{e}_2 = \bar{u}_1 - \bar{e}_1 = \bar{u}_1 - \bar{u}_3 + \bar{u}_2$

$(4) \to (1)$

$$\begin{cases} \bar{e}_1 = -\bar{u}_2 + \bar{u}_3 = (0,-1,1)_{B'} \\ \bar{e}_2 = \bar{u}_1 + \bar{u}_2 - \bar{u}_3 = (1,1,-1)_{B'} \\ \bar{e}_3 = -\bar{u}_1 + \bar{u}_3 = (-1,0,1)_{B'} \end{cases}$$

$$C^{-1} = \begin{array}{c} \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \\ \bar{e}_1 \ \bar{e}_2 \ \bar{e}_3 \\ \text{in } B' \end{array}$$

$$F_B = \underbrace{\overbrace{\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}}^{C} \overbrace{\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}^{F_{B'}}}_{\begin{pmatrix} -1 & 0 & 1 \\ -1 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix}} \overbrace{\begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}}^{C^{-1}}$$

$$F_B = \begin{array}{c} \begin{pmatrix} 1 & -2 & 2 \\ 2 & -3 & 2 \\ 2 & -2 & 1 \end{pmatrix} \\ f(\bar{e}_1)\, f(\bar{e}_2)\, f(\bar{e}_3) \end{array}$$

Another way of getting F_B.

$$S(-1) = \mathcal{L}\left\{\overbrace{(1,1,0)}^{\bar{u}_1}, \overbrace{(0,1,1)}^{\bar{u}_2}\right\} \qquad \begin{cases} f(\bar{u}_1) = -\bar{u}_1 \\ f(\bar{u}_2) = -\bar{u}_2 \end{cases}$$

$$S(1) = \mathcal{L}\left\{\underbrace{(1,1,1)}_{\bar{u}_3}\right\} \qquad \left\{ f(\bar{u}_3) = \bar{u}_3 \right.$$

$$f(\bar{e}_1 + \bar{e}_2) = -\bar{e}_1 - \bar{e}_2 \longrightarrow f(\bar{e}_1) + f(\bar{e}_2) = -\bar{e}_1 - \bar{e}_2 \quad (i)$$

$$f(\bar{e}_2 + \bar{e}_3) = -\bar{e}_2 - \bar{e}_3 \longrightarrow f(\bar{e}_2) + f(\bar{e}_3) = -\bar{e}_2 - \bar{e}_3 \quad (ii)$$

$$f(\bar{e}_1 + \bar{e}_2 + \bar{e}_3) = \bar{e}_1 + \bar{e}_2 + \bar{e}_3 \longrightarrow f(\bar{e}_1) + f(\bar{e}_2) + f(\bar{e}_3) = \bar{e}_1 + \bar{e}_2 + \bar{e}_3 \quad (iii)$$

$$(iii) - (i) : f(\bar{e}_3) = 2\bar{e}_1 + 2\bar{e}_2 + \bar{e}_3 = (2,2,1)_B$$

$$(iii) - (ii) : f(\bar{e}_1) = \bar{e}_1 + 2\bar{e}_2 + 2\bar{e}_3 = (1,2,2)_B \quad (iv)$$

$$(iv) \rightarrow (i) : f(\bar{e}_2) = -\bar{e}_1 - \bar{e}_2 - \underbrace{f(\bar{e}_1)}_{\bar{e}_1 + 2\bar{e}_2 + 2\bar{e}_3} = -2\bar{e}_1 - 3\bar{e}_2 - 2\bar{e}_3 = (-2,-3,-2)_B$$

$$F_B = \begin{array}{c} \begin{pmatrix} 1 & -2 & 2 \\ 2 & -3 & 2 \\ 2 & -2 & 1 \end{pmatrix} \\ \begin{array}{ccc} f(\bar{e}_1) & f(\bar{e}_2) & f(\bar{e}_3) \end{array} \end{array}$$

F_B^{102}?

$$F_{B'} = C^{-1} F_B C \longrightarrow C F_{B'} C^{-1} = \underbrace{\cancel{C C^{-1}}}_{I} F_B \underbrace{\cancel{C C^{-1}}}_{I} \longrightarrow F_B = C F_{B'} C^{-1}$$

$$F_B = C F_{B'} C^{-1} \xrightarrow{\text{so}} F_B^{102} = (C F_{B'} C^{-1})^{102}$$

$$F_B^{102} = C F_{B'} \underbrace{\cancel{C^{-1} C}}_{I} F_{B'} \underbrace{\cancel{C^{-1} C}}_{I} F_{B'} \underbrace{\cancel{C^{-1} C}}_{I} F_{B'} \underbrace{\cancel{C^{-1} C}}_{I} F_{B'} \underbrace{\cancel{C^{-1}}}_{I} \ldots \underbrace{\cancel{C}}_{I} F_{B'} C^{-1} =$$

(102 times

$$= C F_{B'}^{102} C^{-1} = C \underbrace{\begin{pmatrix} (-1)^{102} & 0 & 0 \\ 0 & (-1)^{102} & 0 \\ 0 & 0 & (1)^{102} \end{pmatrix}}_{F_{B'}^{102}} C^{-1} = C \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{I} C^{-1} = C I C^{-1} = C C^{-1} = I$$

$$F_B^{102} = I$$

Eigenvalues 3

Given an automorphism in $\mathbb{P}_2$ the vector space of 2nd degree polynoms named f.
1. Obtain its kernel and image.
2. Knowing that x+1 transforms into itself, that 3 transforms into -3 and that the determinant of the matrix is 6, obtain the eigenvalues of the automorphism.
3. knowing that $L\{(x^2+x+1)\}$ is an eigenspace, obtain F_B.

Note: You cannot use information of later points of the problem in previous points.

1. Automorphism $\longrightarrow$ ingective + suprayective

ingective $\rightarrow$ $Ker(f) = \{\bar{0}\}$ $\rightarrow$ $\dim(Ker(f)) = 0$

suprayective $\rightarrow$ $Im(f) = \mathbb{P}_2$ $\rightarrow$ $\dim(Im(f)) = \dim(\mathbb{P}_2) = 3$

2. $f(x+1) = x+1 \longrightarrow x+1 \in S(1)$

$f(3) = -3 \longrightarrow 3 \in S(-1)$

$|F_B| = 1 \cdot (-1) \cdot \lambda_3 = 6 \longrightarrow \lambda_3 = -6$

Eigenvalues $\begin{cases} \lambda_1 = 1 \\ \lambda_2 = -1 \\ \lambda_3 = -6 \end{cases}$